Hidden symmetries in the asymmetric exclusion process

O. Golinelli, K. Mallick

Service de Physique Théorique,
Cea Saclay, 91191 Gif-sur-Yvette, France

Abstract

We present a spectral study of the evolution matrix of the totally asymmetric exclusion process on a ring at half filling. The natural symmetries (translation, charge conjugation combined with reflection) predict only two fold degeneracies. However, we have found that degeneracies of higher order also exist and, as the system size increases, higher and higher orders appear. These degeneracies become generic in the limit of very large systems. This behaviour can be explained by the Bethe Ansatz and suggests the presence of hidden symmetries in the model.

Keywords: ASEP, Markov matrix, symmetries, spectral degeneracies, Bethe Ansatz.

1 Introduction

The asymmetric simple exclusion process (ASEP) that plays the role of a paradigm in non-equilibrium statistical physics is a model of driven diffusive particles on a lattice interacting through hard-core exclusion (Katz, Lebowitz and Spohn 1984). Introduced originally as a model for protein synthesis on RNA (MacDonald and Gibbs 1969) and for hopping conductivity (Richards 1977) in solid electrolytes, the ASEP has found a wide spectrum of applications ranging from traffic flow (Chowdhury *et al.* 2000) to surface growth

1

(Halpin-Healy and Zhang 1995, Krug 1997) and sequence alignment (Bundschuh 2002). The ASEP has also motivated many theoretical studies, providing a 'minimal' model sophisticated enough to yield a rich phenomenology but elementary enough to allow exact solutions [for a review see (Derrida 1998) and (Schutz 2001)].

The ASEP is a memoryless stochastic process governed by simple local update rules that can be embodied in a Markov evolution matrix M. This matrix contains all the dynamical information about the system: for example, the eigenvector of M with eigenvalue 0 represents the steady state of the ASEP and embodies the stationary correlation functions. In particular, the matrix product representation of the stationary state (Derrida *et al.* 1993) allows to calculate steady state properties and large deviation functionals (Derrida *et al.* 2003). More generally, the spectrum of M encodes the relaxation behaviour of the ASEP and all non-stationary correlation functions. The Markov matrix M can be diagonalized with the Bethe Ansatz that allows to calculate the energy gap and the dynamical exponent of the ASEP (Dhar 1987, Gwa and Spohn 1992, Kim 1995, Golinelli and Mallick 2004).

Although the steady state of the ASEP and the lowest excitations have been extensively studied, little effort has been devoted to investigate global spectral properties of the Markov matrix (Bilstein and Wehefritz 1997, Dudziński and Schütz 2000, Nagy *et al.* 2002). In this work, we focus on the degeneracies in the spectrum of the Markov matrix for the totally asymmetric simple exclusion process (TASEP) on a periodic ring at half filling (*i.e.*, the density of particles is one-half). The exclusion process has some natural invariance properties by translation, charge conjugation and reflection. At half filling these symmetries imply the existence of doublets in the spectrum of M. However, we find that multiplets with degeneracies of higher order also exist for the TASEP. Moreover, the number of these multiplets and their degeneracies are given by simple combinatorial factors. These remarkable spectral properties suggest that the TASEP at half filling possesses some hidden symmetries other than the obvious ones.

This article is organized as follows. In section 2, we recall the definition of the ASEP on a periodic ring and discuss the translation, the charge conjugation and the reflection symmetries, and their effect on the spectrum of the Markov matrix M. In section 3, we present some numerical evidence for spectral degeneracies in the TASEP at half filling. We show that the combinatorial factors involved can be understood, heuristically, from the Bethe Ansatz. In Section 4, we present some concluding remarks and discuss spectral degeneracies in closely related models. In the Appendix, we analyze the natural symmetries of the ASEP with a tagged particle.

2 Natural symmetries of the ASEP on a ring

2.1 The model

We consider the exclusion process on a periodic one dimensional lattice with L sites (sites i and $L + i$ are identical) and n particles. A lattice site cannot be occupied by more than one particle (*exclusion rule*). The state of a site i ($1 \le i \le L$) can thus be characterized by the Boolean number $\tau_i = 0, 1$ according as the site i is empty or occupied.

The system evolves with time according to the following stochastic rule: a particle on a site i at time t jumps, in the interval between t and $t+dt$, with probability $p\, dt$ to the neighbouring site $i + 1$ if this site is empty and with probability $q\, dt$ to the site $i - 1$ if this site is empty. The jump rates p and q are normalized such that $p + q = 1$. In the totally asymmetric exclusion process (TASEP) the jumps are totally biased in one direction ($p = 1$ and $q = 0$).

The number n of particles is conserved by the dynamics. The total number of configurations for n particles on a ring with L sites is given by $\Omega = L!/[n!(L - n)!]$. A configuration $\mathcal{C}$ can be represented by the sequence $(\tau_1, \tau_2, \ldots, \tau_L)$. We call $\psi_t(\mathcal{C})$ the probability of configuration $\mathcal{C}$ at time t. As the exclusion process is a continuous-time Markov process, the time evolution of $\psi_t(\mathcal{C})$ is determined by the master equation

$$\frac{d}{dt}\psi_t(\mathcal{C}) = \sum_{\mathcal{C}'} M(\mathcal{C}, \mathcal{C}')\psi_t(\mathcal{C}')\,. \tag{1}$$

The Markov matrix M encodes the dynamics of the exclusion process: the element $M(\mathcal{C}, \mathcal{C}')$ is the transition rate from configuration $\mathcal{C}'$ to $\mathcal{C}$ and the diagonal term $M(\mathcal{C}, \mathcal{C}) = -\sum_{\mathcal{C}'} M(\mathcal{C}', \mathcal{C})$ represents the exit rate from configuration $\mathcal{C}$.

A right eigenvector ψ is associated with the eigenvalue E of M if

$$M\psi = E\psi\,. \tag{2}$$

The matrix M is a real non-symmetric matrix and, therefore, its eigenvalues (and eigenvectors) are either real numbers or complex conjugate pairs. The spectrum of M contains the eigenvalue $E = 0$ and the associated right eigenvector is the stationary state. For the ASEP on a ring the steady state is uniform: the stationary probability of any configuration $\mathcal{C}$ is given by $\psi(\mathcal{C}) = 1/\Omega$. Because the dynamics is ergodic (*i.e.*, M is irreducible and aperiodic), the Perron-Frobenius theorem (see, for example, Gantmacher 1964) implies that 0 is a non-degenerate eigenvalue and that all other eigenvalues

E have a strictly negative real part; the relaxation time of the corresponding eigenmode is $\tau = -1/\mathrm{Re}(E)$.

Remark: A configuration can also be characterized by the positions of the n particles on the ring, $(x_1, x_2, \ldots, x_n)$ with $1 \le x_1 < x_2 < \ldots < x_n \le L$. With this representation, the eigenvalue equation (2) becomes

$$
\begin{aligned}
E\psi(x_1, \ldots, x_n) = \\
\sum_i p\left[\psi(x_1, \ldots, x_{i-1},\ x_i - 1,\ x_{i+1}, \ldots, x_n) - \psi(x_1, \ldots, x_n)\right] + \\
\sum_j q\left[\psi(x_1, \ldots, x_{j-1},\ x_j + 1,\ x_{j+1}, \ldots, x_n) - \psi(x_1, \ldots, x_n)\right],
\end{aligned}
\tag{3}
$$

where the sum runs over the indexes i such that $x_{i-1} < x_i - 1$ and over the indexes j such that $x_j + 1 < x_{j+1}$, *i.e.*, such that the corresponding jumps are allowed.

2.2   Translation invariance

The translation operator T shifts simultaneously all the particles one site forward:

$$
T(\tau_1, \tau_2, \ldots, \tau_L) = (\tau_L, \tau_1, \tau_2, \ldots, \tau_{L-1}).
\tag{4}
$$

Because of periodicity we have $T^L = 1$. Thus, the eigenvalues (or impulsions) k of T are simply the L-th roots of unity:

$$
k = e^{2i\pi m/L} \quad \text{for } m = 0, \ldots, L-1.
\tag{5}
$$

We denote by $\mathbf{T}(k)$ the eigenspace of T corresponding to the impulsion k. The projection operator $\mathcal{T}_k$ over this eigenspace is given by

$$
\mathcal{T}_k = \frac{1}{L} \sum_{r=0}^{L-1} k^{-r} T^r.
\tag{6}
$$

The complex conjugation transforms a T-eigenvector with eigenvalue k into an eigenvector with eigenvalue $k^* = k^{-1}$, *i.e.*, $\mathbf{T}(k^*) = [\mathbf{T}(k)]^*$.

The ASEP on a periodic lattice is translation invariant, *i.e.*,

$$
MT = TM.
\tag{7}
$$

The matrix M and the translation operator T can therefore be simultaneously diagonalized: M leaves each eigenspace $\mathbf{T}(k)$ invariant. We denote by $\mathbf{Sp}(k)$ the set of the eigenvalues of M restricted to $\mathbf{T}(k)$. Using complex conjugation we obtain the property

$$
\mathbf{Sp}(k^*) = [\mathbf{Sp}(k)]^*.
\tag{8}
$$

2.3 Right-Left Reflection

The reflection operator R reverses the right and the left and is defined by

$$R(\tau_1, \tau_2, \ldots, \tau_L) = (\tau_L, \tau_{L-1}, \ldots, \tau_2, \tau_1). \tag{9}$$

We have $R^2 = 1$; the eigenvalues of R are thus $r = \pm 1$. The reflection R reverses the translations, *i.e.*,

$$RT = T^{-1}R, \tag{10}$$

and transforms a T-eigenvector with eigenvalue k into an eigenvector with eigenvalue $k^* = k$. The operator R does not commute with the Markov matrix M because the asymmetric jump rates $(p \neq q)$ are not invariant under the exchange of right and left. Writing explicitly the dependence of M on the jump rates, we have

$$RM_{(p,q)}R^{-1} = M_{(q,p)}. \tag{11}$$

Thus, in general the reflection is *not* a symmetry of the ASEP (it is only a symmetry of the symmetric exclusion process).

2.4 Charge conjugation

The charge conjugation operator C exchanges particles and holes in the system, *i.e.*, a configuration with n particles is mapped into a configuration with $L - n$ particles:

$$C(\tau_1, \tau_2, \ldots, \tau_L) = (1 - \tau_1, 1 - \tau_2, \ldots, 1 - \tau_L). \tag{12}$$

The operator C satisfies the relations:

$$C^2 = 1, \quad CT = TC, \quad CR = RC. \tag{13}$$

By charge conjugation, particles jumping forward are mapped into holes jumping forward. But holes jumping forward are equivalent to particles jumping backward. Writing explicitly the dependence of M on the number of particles and on the jump rates, we have

$$CM_{(n,p,q)}C^{-1} = M_{(L-n,q,p)}. \tag{14}$$

We notice that the number of particles is conserved by C only at half filling $(L = 2n)$. Thus, the charge conjugation is *not* a symmetry of the ASEP except for the symmetric $(p = q)$ exclusion process at half filling.

2.5 CR symmetry

Using equations (11) and (14) to combine the charge conjugation C with the reflection R, we obtain

$$(CR)M_{(n,p,q)}(CR)^{-1} = M_{(L-n,p,q)}. \tag{15}$$

Thus, for a given L, the CR operator maps the ASEP with n particles into the ASEP with the same jumping rates but with $L - n$ particles. This implies that the spectrum of M for n particles is identical with the spectrum for $L - n$ particles (because CR transforms eigenvectors of M_n into eigenvectors of M_{L-n}).

Hereafter, we shall always consider the ASEP model at half filling, *i.e.*, $L = 2n$. In this case, the CR operator constitutes an exact symmetry because $(CR)\,M = M\,(CR)$.

The ASEP at half filling is therefore invariant under each of the two symmetries, translation T and CR. Note that they do not commute with each other. Rather, we obtain from equations (10) and (13)

$$(CR)T = T^{-1}(CR). \tag{16}$$

Hence, the CR transformation maps the subspace $\mathbf{T}(k)$ into $\mathbf{T}(k^*)$ and thus

$$\mathbf{Sp}(k) = \mathbf{Sp}(k^*). \tag{17}$$

For $k \neq \pm 1$, the two subspaces $\mathbf{T}(k)$ and $\mathbf{T}(k^*)$ are distinct and equation (17) implies that the corresponding eigenvalues of M are doubly degenerate. For $k = \pm 1$, the CR transformation leaves $\mathbf{T}(k)$ invariant. Thus, the natural symmetries do not predict any degeneracies in the sets $\mathbf{Sp}(\pm 1)$. However, each of the subspaces $\mathbf{T}(\pm 1)$ is split into two smaller subspaces invariant under CR and on which $CR = \pm 1$.

Equations (8) and (17) imply that for all k the set $\mathbf{Sp}(k)$ is self-conjugate, *i.e.*,

$$\mathbf{Sp}(k) = [\mathbf{Sp}(k)]^*. \tag{18}$$

This means that $\mathbf{Sp}(k)$, for all k, is made only of real numbers or complex conjugate pairs.

In the Appendix, we explain how a charge conjugation operator C can be defined for the ASEP with a *tagged* particle so that CR remains a symmetry.

3   Spectral degeneracies in the TASEP

The above discussion, based on obvious symmetries of the ASEP at half filling with generic jump rates p and q, suggests that the spectrum of the

Markov matrix should be composed of singlets (for impulsion $k = \pm 1$) and doublets (for $k \neq \pm 1$); so multiplets of higher order have no reason to appear and therefore should not exist generically.

In this section we describe spectra for systems of size $L \leq 18$ at half filling obtained from numerical diagonalization for the TASEP, i.e. the model where jumps are allowed only in one direction ($q = 0$). These spectra contain many unexpected degeneracies of very special orders. Our numerical observations lead us to conjecture some formulae for general L. We shall show that these formulae can be derived, at least heuristically, from the Bethe Ansatz equations. The existence of such degeneracies of higher orders strongly suggests the presence of unknown underlying symmetries.

3.1 Numerical results and conjectures

The spectral degeneracies of the Markov matrix M are numerically calculated with the following procedure. First, the matrix M is built for a given size L and number of particles n. As the system is periodic, we use the translation invariance T in order to split the space of configurations into L subspaces $\mathbf{T}(k)$ with impulsion k (see section 2.2). Thus M is block-diagonal with L blocks. The spectrum of each block is then computed by a routine of the Lapack library (Anderson et al., 1999) with the standard numerical double precision (around 15 decimal digits). The L spectra obtained by numerical diagonalization are then collected together in order to search and count the degeneracies.

The time required to compute the complete spectrum of a matrix of size N is proportional to N^3. Since the sizes of the blocks $\mathbf{T}(k)$ grow as 2^L, the computation time grows as 8^L. Thus only small systems can be studied by this method: we studied systems of size $L \leq 18$.

The problem is now to choose a criterion to decide when two eigenvalues, computed with finite precision, are equal. By calculating the differences between all the eigenvalues we observe that these differences are always either smaller than 10^{-12} or larger than 10^{-4}. Consequently we deduce that two eigenvalues with a difference smaller than 10^{-12} are in fact equal and that this apparent discrepancy is due to the finite precision of the computation. This allows us to enumerate the degeneracies. We emphasize that the existence of this gap of eight orders of magnitude (between 10^{-12} and 10^{-4}) in the data of the differences is crucial. If the distribution of the differences was more continuous (*i.e.*, if there was no obvious gap), the choice of a criterion to decide which eigenvalues are identical would have been much more difficult and somehow arbitrary.

Let $m(d)$ be the number of multiplets with degeneracy of order d, *i.e.*, the

L	Ω	$d=1$	$d=2$	$d=6$	$d=20$	$d=70$
2	2	2				
4	6	4	1			
6	20	8	6			
8	70	16	24	1		
10	252	32	80	10		
12	924	64	240	60	1	
14	3432	128	672	280	14	
16	12870	256	1792	1120	112	1
18	48620	512	4608	4032	672	18

Table 1: *Spectral degeneracies in the TASEP at half filling: L is the size of the lattice; Ω is the dimension of the configuration space; the other columns give $m(d)$ the number of multiplets of degeneracy d. These numbers are given by the simple combinatorial formulae (19) and (20).*

number of sets of d equal eigenvalues. By definition, we have $\sum_d d\, m(d) = \Omega$, the total number of eigenvalues. For the TASEP model with half-filling, the numbers $m(d)$ are given in Table 1 for sizes $L \le 18$. Although only singlets ($d = 1$) and doublets ($d = 2$) are predicted by the natural symmetries T and CR, numerous degeneracies of higher order also appear. We notice from Table 1 that the observed orders of degeneracies are only 1,2,6,20,70. These numbers obey the sequence

$$d_r = \begin{pmatrix} 2r \\ r \end{pmatrix} = \frac{(2r)!}{r!\, r!} \; ; \tag{19}$$

for a system of size $2n$, r takes all integral values from 0 to $n/2$. From an empirical study of Table 1 we also remark that for $L = 2n \le 18$,

$$m(d_r) = \begin{pmatrix} n \\ 2r \end{pmatrix} 2^{n-2r} \, . \tag{20}$$

For $r = 0$ and 1, these equations provide the number of singlets $m(1) = 2^n$ and the number of doublets $m(2) = n(n-1)2^{n-3}$. We conjecture that equations (19) and (20) are true for all L. We have verified that these relations satisfy the sum rule $\sum_d d\, m(d) = \Omega$:

$$\sum_{r \ge 0} d_r\, m(d_r) = \sum_{0 \le r \le n/2} \frac{2^{n-2r}\, n!}{(n-2r)!\, r!\, r!} = \begin{pmatrix} 2n \\ n \end{pmatrix} = \Omega \, . \tag{21}$$

The last equality is obtained by identifying the terms of order x^0 in the

identity

$$\left(2 + x + \frac{1}{x}\right)^n = \sum_{n_1+n_2+n_3=n} \frac{2^{n_1} x^{n_2-n_3} n!}{n_1! \, n_2! \, n_3!} = \frac{(x+1)^{2n}}{x^n} = \sum_k \binom{2n}{k} x^{k-n} . \quad (22)$$

Using equations (19) and (20), we can analyze the large L (or n) behaviour. For example, we note that the number of singlets, 2^n, becomes negligible with respect to the total number of eigenvalues, $\Omega \sim 4^n/n^{1/2}$, as L grows. As explained in section 2.5, the CR symmetry implies that the singlets necessarily belong to the subspaces $\mathbf{T}(\pm 1)$. However, the number of these singlets is much smaller than the dimension of subspaces $\mathbf{T}(\pm 1)$ (which is approximately given by $\Omega/n \propto 4^n/n^{3/2}$). Therefore, multiplets must necessarily exist in the subspaces $\mathbf{T}(\pm 1)$: this fact is an indication for a hidden symmetry that would commute with T but not commute with CR, for example.

More precisely, the number $m(d)$ of multiplets is maximal for $r \sim n/6$ which corresponds to a degeneracy of order $d \propto 2^{n/3}$ with $m(d) \propto 3^n$. Then, we obtain that $d\,m(d) \propto (2^{1/3}3)^n \ll \Omega$ because $2^{1/3}3 \approx 3.78 < 4$. On the other hand, the number $d\,m(d)$ of eigenvalues counted with their degeneracies is maximal for $r \sim n/4$. In this case, we have $d \propto 2^{n/2}$, $m(d) \propto 8^{n/2}$ and $d\,m(d) \propto 4^n \propto \Omega$. Loosely speaking, we deduce from this estimation that most of the eigenvalues in the spectrum are degenerate with a degeneracy of order $2^{n/2}$.

3.2 Bethe Ansatz analysis of the spectral degeneracies

The Markov matrix of the exclusion process can be diagonalized thanks to the *Bethe Ansatz* (Dhar, 1987) that assumes that an eigenvector ψ of M can be written as

$$\psi(x_1, \ldots, x_n) = \sum_{\sigma \in \Sigma_n} A_\sigma \, z_{\sigma(1)}^{x_1} \, z_{\sigma(2)}^{x_2} \ldots z_{\sigma(n)}^{x_n} , \quad (23)$$

where Σ_n is the group of the $n!$ permutations of n indexes. The coefficients $\{A_\sigma\}$ and the wave-numbers $\{z_1, \ldots, z_n\}$ are complex numbers to be determined. For TASEP at half filling, it is convenient to use the fugacity variables $Z_i = 2/z_i - 1$ and to introduce an auxiliary complex variable Y. The Bethe equations can then be reformulated as explained below. (For more details, see Gwa and Spohn 1992; Golinelli and Mallick 2004). In terms of the n-th roots of Y defined as

$$y_m = Y^{1/n} e^{(m-1)2i\pi/n} \quad \text{for } m = 1, \ldots, n , \quad (24)$$

we calculate $2n$ numbers $(Z_1, \ldots, Z_{2n})$ given by

$$Z_m = (1 - y_m)^{1/2}; \quad Z_{m+n} = -Z_m. \tag{25}$$

In order to select n fugacities among $(Z_1, \ldots, Z_{2n})$, we introduce a choice function $c : \{1, \ldots, n\} \to \{1, \ldots, 2n\}$ that satisfies $1 \leq c(1) < \ldots < c(n) \leq 2n$. The Bethe equations now become equivalent to the *self-consistency* equation

$$A_c(Y) = Y, \tag{26}$$

$$\text{with} \quad A_c(Y) = -4^n \prod_{j=1}^{n} \frac{Z_{c(j)} - 1}{Z_{c(j)} + 1}. \tag{27}$$

Given the choice function c and a solution Y of this equation, the $Z_{c(j)}$'s are determined from Eq. (25) and the corresponding eigenvalue E_c is given by

$$2E_c = -n + \sum_{j=1}^{n} Z_{c(j)}. \tag{28}$$

We now assume that the Bethe Ansatz is complete, *i.e.*, the Bethe equations yield a complete basis of eigenvectors for the ASEP. Besides, we also note that the number of different choice functions is $\Omega = (2n)!/n!^2$, Ω being precisely the size of the Markov matrix. We are thus led to make the stronger hypothesis that for each choice function c (among the Ω possible choice functions), the self-consistency equation (26) has a unique solution Y that yields one eigenvector ψ_c and one eigenvalue E_c. We further assume that these eigenvectors are linearly independent.

Using this one-to-one hypothesis, we will now explain the spectral degeneracies and derive equations (19) and (20). We recall that the set $(Z_1, \ldots, Z_{2n})$ is constituted of n pairs, each pair containing two opposite fugacities ($Z_{m+n} = -Z_m$). A choice function c selects n numbers in the set $(Z_1, \ldots, Z_{2n})$; amongst these n selected fugacities, $2r$ appear in pairs and $n - 2r$ appear as singles, with $0 \leq 2r \leq n$. The crucial remark is that for a given choice function c the Bethe equations and the eigenvalue E_c depend only on the $n - 2r$ selected single fugacities. This fact is clear from equations (26, 27 and 28) because contributions of Z_m and Z_{m+n} cancel each other. Thus two different choice functions c_1 and c_2 that contain the same single fugacities (but different pairs) lead to the same eigenvalue but to different eigenvectors. Hence all the choice functions that contain the same single fugacities form a multiplet with the same eigenvalue. The number of these choice functions is, thus, the order of degeneracy of the corresponding multiplet. This order of degeneracy is determined solely by the integer r.

For a given value of r (with $0 \leq 2r \leq n$), we now calculate (i) the number d_r of choice functions in each multiplet, *i.e.*, its degeneracy and (ii) the number $m(d_r)$ of different multiplets.

(i) Two choice functions c and c' are in the same multiplet if they differ only in the choice of the r pairs. The $n-2r$ single fugacities having been fixed, we must choose r pairs amongst $2r$ available pairs. The number of choice functions in a multiplet is therefore given by $d_r = \begin{pmatrix} 2r \\ r \end{pmatrix}$ in agreement with equation (19).

(ii) A multiplet is characterized by its $s = n - 2r$ single fugacities. The number of different ways to select these single fugacities is given by

$$m(d_r) = \frac{2n(2n-2)\dots(2n-2(s-1))}{s!} = \begin{pmatrix} n \\ s \end{pmatrix} 2^s = \begin{pmatrix} n \\ 2r \end{pmatrix} 2^{n-2r}, \quad (29)$$

in agreement with equation (20).

These calculations thus explain the numerical results described in the previous section. However, our analysis is based on the hypothesis of a one-to-one correspondence between choice functions and linearly independent eigenvectors. We emphasize that this one-to-one hypothesis is more stringent than the assumption of the completeness of the Bethe Ansatz. At the present stage, we do not have a proof of this hypothesis but we have tested it numerically for small system sizes: it is verified for $L = 2, 4, 6, 8$.

4   Conclusion and Generalizations

In this work, we have studied the degeneracies in the spectrum of the Markov matrix of the totally asymmetric exclusion process on a periodic ring at half filling. Numerical results show the presence of many degeneracies with specific orders. These orders and the number of multiplets of a given degeneracy can be calculated from the Bethe Ansatz equations under the assumption of a one-to-one correspondence between the eigenvectors of the Markov matrix and some choice functions of the fugacities that appear in the Bethe Ansatz. Certainly, a proof (or a disproof) of this hypothesis would be of interest. We believe that this assumption must be true, at least in a weaker form, because it leads to a correct enumeration of the spectral degeneracies and multiplets.

We emphasize that the observed higher order degeneracies are not predicted by the natural symmetries of the TASEP. Indeed, we showed that invariance under translation and charge conjugation plus reflection, predicts only the existence of two fold degeneracies in the spectrum. Therefore, we

believe that the ASEP presents some hidden symmetries, that are responsible for the numerous and large generic degeneracies in the spectrum of the Markov matrix. By inspecting numerically the multiplets, we have observed that each of them are composed of eigenvalues with different impulsion k: thus a hidden symmetry does not necessarily commute with the translation operator T. A better understanding of these symmetries would be of great significance.

We have also analyzed the degeneracies in some related models in order to study the influence of asymmetry and periodicity. For the *partially* asymmetric exclusion process with non zero left and right jump rates ($p > q > 0$) on a periodic lattice and at half filling, numerical computations show the existence of singlets and doublets and the absence of degeneracies of higher order. However, the number of singlets is exactly 2^n (*i.e.*, the same as for the TASEP). From the natural symmetries, one would expect a much larger number of singlets (of the order of $4^n/n^{3/2}$). This is again a hint for some hidden symmetries in the partially asymmetric exclusion process (Alcaraz *et al.* 1994). The TASEP can be also defined on an open lattice: it presents a rich phenomenological behaviour that was thoroughly investigated using the matrix product representation (Derrida *et al.* 1993). In this system the translation symmetry T is lost and the number of particles is not conserved. We have numerically computed spectra for different sizes L and different entrance and exit rates: no degeneracies have been observed in any of these cases.

Another direction of interest is to study the TASEP model for *arbitrary* filling. In fact, we have numerically observed degeneracies of high order in the spectra when the density is a simple fraction. Degeneracies at arbitrary filling can be analyzed by Bethe Ansatz but the calculations seem to be far more complicated. General results will be presented in a future work (Golinelli and Mallick 2004a).

Acknowledgments

Appendix : Natural symmetries of the ASEP with a tagged particle

To study the motion of particles in the ASEP, it is often interesting to tag one of them (Spohn, 1991). All particles are then automatically labelled

because overtaking is forbidden. A configuration $\mathcal{C}$ of n particles on a ring with L sites can now be specified by the sequence $(\tau_1, \tau_2, \ldots, \tau_L; m)$ where $\tau_i = 0, 1$ according as the site i is empty or occupied and the tag is on the m-th occupied site with $1 \le m \le n$ (m is defined modulo n). The number of configurations of the ASEP with a tagged particle is given by $n\,\Omega = n\,L!/[n!(L-n)!]$.

The translation operator T shifts simultaneously all the particles one site forward, *i.e.*

$$T(\tau_1, \tau_2, \ldots, \tau_L; m) = (\tau_L, \tau_1, \ldots, \tau_{L-1}; m + \tau_L). \tag{30}$$

Indeed, the index m increases by 1 if and only if T moves a particle from site L to site 1. The ASEP with a tagged particle remains translation invariant: let M be its Markov matrix, we have $MT = TM$. As $T^L = 1$, the eigenvalues k of T are the L-th roots of unity [see Eq. (5)].

The ASEP with a tagged particle presents a new symmetry P, which consists in shifting the tag one particle forward but without moving the particles, *i.e.*

$$P(\tau_1, \ldots, \tau_L; m) = (\tau_1, \ldots, \tau_L; m + 1). \tag{31}$$

As $P^n = 1$, the eigenvalues α of the tag-shift operator P are simply the n-th roots of unity. We have

$$PM = MP; \quad TP = PT. \tag{32}$$

The matrix M can thus be simultaneously diagonalized with T and P. We call $\mathbf{E}(k, \alpha)$ the common eigenspace of T and P corresponding to the eigenvalues k and α respectively, where

$$
\begin{aligned}
k &= e^{2i\pi j/L} &\text{for } j = 0, \ldots, L-1, \tag{33} \\
\alpha &= e^{2i\pi a/n} &\text{for } a = 0, \ldots, n-1. \tag{34}
\end{aligned}
$$

and $\mathbf{Sp}(k, \alpha)$ the set of the eigenvalues of M restricted to $\mathbf{E}(k, \alpha)$.

We remark that the Markov matrix of the ASEP *without* a tag is the restriction of the Markov matrix of the ASEP with a tag to the subspace $\alpha = 1$. Indeed, the projection on the subspace $\alpha = 1$ of a configuration $(\tau_1, \tau_2, \ldots, \tau_L; m)$ is given by $\sum_{j=1}^{n}(\tau_1, \tau_2, \ldots, \tau_L; j)/n$; this projection renders the particles indistinguishable.

Let us now consider a configuration $(\tau_1, \tau_2, \ldots, \tau_L; m)$, *i.e.*, the tagged particle is the m-th particle from the left. With the left-right reflection R, it becomes the m-th one from the *right*, so the $(n + 1 - m)$-th one from the *left*, *i.e.*,

$$R(\tau_1, \tau_2, \ldots, \tau_L; m) = (\tau_L, \ldots, \tau_2, \tau_1; n + 1 - m). \tag{35}$$

We recall that R is not a symmetry of the ASEP because the right and left jump rates, p and q, are swapped.

The action of the charge conjugation operator C is more subtle. The operator C transforms a system with n particles (including one tagged particle) into one with $L - n$ particles (including one tagged particle). The dimensions of the corresponding configuration spaces will be the same only in the half filling case; thus for the tagged ASEP, the charge conjugation C can be well defined only for $L = 2n$. Moreover, because C transforms the tagged particle into a hole, the tag cannot remain on the same site and we must specify where it goes. One solution, that will ensure the CR-invariance of the ASEP, is to define C as follows:

$$C(\tau_1, \tau_2, \ldots, \tau_L;\ m) = (1 - \tau_1, 1 - \tau_2, \ldots, 1 - \tau_L;\ n + 1 - m)\,. \qquad (36)$$

Hence, after charge conjugation, the tag is on the m-th particle from the *right*. Consequently, we obtain, at half filling

$$CR(\tau_1, \tau_2, \ldots, \tau_L; m) = (1 - \tau_L, \ldots, 1 - \tau_2, 1 - \tau_1; m). \qquad (37)$$

It follows that

$$(CR)^2 = 1\ ;\quad P(CR) = (CR)P\,. \qquad (38)$$

It is also possible to show that CR is a symmetry of the model:

$$M(CR) = (CR)M\,. \qquad (39)$$

(a little care is needed when analyzing jumps between sites L and 1).

Combining Eq. (30) with Eq. (37), the relation between CR and T is obtained

$$CRT(\tau_1, \tau_2, \ldots, \tau_L; m) = (1 - \tau_{L-1}, \ldots, 1 - \tau_1, 1 - \tau_L; m + \tau_L)\,, \qquad (40)$$
$$TCRT(\tau_1, \tau_2, \ldots, \tau_L; m) = (1 - \tau_L, 1 - \tau_{L-1}, \ldots, 1 - \tau_1; m + 1)\,, \qquad (41)$$

so $TCRT = PCR$, or equivalently

$$T\,(CR) = (CR)\,PT^{-1}\,. \qquad (42)$$

Hence the CR transformation maps the subspace $\mathbf{E}(k, \alpha)$ into $\mathbf{E}(\alpha k^*, \alpha)$ and thus

$$\mathbf{Sp}(k, \alpha) = \mathbf{Sp}(\alpha k^*, \alpha). \qquad (43)$$

For $k^2 \neq \alpha$, these two subspaces are distinct but have the same spectrum: *their eigenvalues are doubly degenerate.* For $k^2 = \alpha$, the CR symmetry leaves $\mathbf{E}(k, \alpha)$ invariant and splits it into two smaller subspaces with $cr = \pm 1$,

without degeneracy. With the notations of Eqs. (33, 34), the condition $k^2 = \alpha$ is equivalent to $j = a$ or $a + n$.

Complex conjugation implies that $[\mathbf{Sp}(k, \alpha)]^* = \mathbf{Sp}(k^*, \alpha^*)$. Therefore, we obtain from Eq. (43) that $\mathbf{Sp}(k, \alpha)$ is self-conjugate if $\alpha = 1$ for any k (this case is equivalent to the ASEP without tag) or if $\alpha = -1$ for $k = \pm 1$.

To summarize, the CR symmetry can still be defined at half filling for the generic ASEP with a tagged particle. This symmetry together with translation invariance predicts the existence of doublets in the spectrum [see Eq. (43)].

For the totally asymmetric model (TASEP) with a tagged particle, numerical diagonalization shows numerous degeneracies of order higher than 2 inside each P-invariant subspace. It follows that hidden symmetries should also exist in this system and should commute with the tag-shift operator P.

References

- F. C. Alcaraz, M. Droz, M. Henkel, V. Rittenberg, 1994, *Reaction-diffusion processes, critical dynamics, and quantum chains*, Ann. Phys. **230**, 250.

- E. Anderson and al., 1999, *LAPACK Users' Guide*, (Philadelphia, SIAM)

- U. Bilstein, B. Wehefritz, 1997, *Spectra of non-Hermitian quantum spin chains describing boundary induced phase transitions*, J. Phys. A **30**, 4925.

- R. Bundschuh, 2002, *Asymmetric exclusion process and extremal statistics of random sequences*, Phys. Rev. E **65**, 031911.

- D. Chowdhury, L. Santen, A. Schadschneider, 2000, *Statistical physics of vehicular traffic and some related systems*, Phys. Rep. **329**, 199.

- B. Derrida, 1998, *An exactly soluble non-equilibrium system: the asymmetric simple exclusion process*, Phys. Rep. **301**, 65.

- B. Derrida, M. R. Evans, V. Hakim, V. Pasquier, 1993, *Exact solution of a 1D asymmetric exclusion model using a matrix formulation*, J. Phys. A: Math. Gen. **26**, 1493.

- B. Derrida, J. L. Lebowitz, E. R. Speer, 2003, *Exact large deviation functional of a stationary open driven diffusive system: the asymmetric exclusion process*, J. Stat. Phys. **110**, 775.

- D. Dhar, 1987, *An exactly solved model for interfacial growth*, Phase Transitions **9**, 51.

- M. Dudziński, G. M. Schütz, 2000, *Relaxation spectrum of the asymmetric exclusion process with open boundaries*, J. Phys. A **33**, 8351.

- F.R. Gantmacher, 1964, *Matrix theory*, (Chelsea, New York)

- O. Golinelli, K. Mallick, 2004, *Bethe Ansatz calculation of the spectral gap of the asymmetric exclusion process*, J. Phys. A **37**, 3321.

- O. Golinelli, K. Mallick, 2004a, *Spectral degeneracies in the totally asymmetric exclusion process*, Preprint Spht T04-167.

- L.-H. Gwa, H. Spohn, 1992, *Bethe solution for the dynamical-scaling exponent of the noisy Burgers equation*, Phys. Rev. A **46**, 844.

- T. Halpin-Healy, Y.-C. Zhang, 1995, *Kinetic roughening phenomena, stochastic growth, directed polymers and all that*, Phys. Rep. **254**, 215.

- S. Katz, J. L. Lebowitz, H. Spohn, 1984, *Nonequilibrium steady states of stochastic lattice gas models of fast ionic conductors*, J. Stat. Phys. **34**, 497.

- D. Kim, 1995, *Bethe Ansatz solution for crossover scaling functions of the asymmetric XXZ chain and the Kardar-Parisi-Zhang-type growth model*, Phys. Rev. E **52**, 3512.

- J. Krug, 1997, *Origins of scale invariance in growth processes*, Adv. Phys. **46**, 139.

- C. T. MacDonald, J. H. Gibbs, 1969, *Concerning the kinetics of polypeptide synthesis on polyribosomes*, Biopolymers **7**, 707.

- Z. Nagy, C. Appert, L. Santen, 2002, *Relaxation times in the ASEP model using a DMRG method*, J. Stat. Phys **109**, 623.

- P. M. Richards, 1977, *Theory of one-dimensional hopping conductivity and diffusion*, Phys. Rev. B **16**, 1393.

- G.M. Schutz, 2001, Phase Transitions and Critical Phenomena, Vol. 19, (Academic, London).

- H. Spohn, 1991, *Large scale dynamics of interacting particles*, (Berlin, Springer).